Vector Analysis

Example (1):

If $\overline{R}(t) = \sin t\,\overline{i} + \cos t\,\overline{j} + t\,\overline{k}$ find $\dfrac{dR(t)}{dt}, \quad \dfrac{d^2R(t)}{dt^2}$.

Solution:

$$\frac{d\overline{R}(t)}{dt} = \frac{d}{dt}\left(\sin t\overline{i} + \cos t\overline{j} + \overline{k}\right) = \cos t\ \overline{i} - \sin t\ \overline{j} + \overline{k}$$

$$\therefore \left|\frac{d\overline{R}(t)}{dt}\right| = \sqrt{\cos^2 t + (-\sin t)^2 + 1} = \sqrt{2}$$

$$\frac{d^2\overline{R}(t)}{dt^2} = \frac{d\cos t}{dt}\overline{i} + \frac{d(-\sin t)}{dt}\overline{j} + \frac{d}{dt}(1)\overline{k} = -\sin t\ \overline{i} - \cos t\ \overline{j}$$

$$\therefore \left|\frac{d^2\overline{R}(t)}{dt^2}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1$$

Example (2):

If $\overline{A} = 5t^2\ \overline{i} + t\ \overline{j} - t^3\ \overline{k}, \qquad \overline{B} = \sin t\ \overline{i} + \cos t\ \overline{j}$

Find $\quad (i)\ \dfrac{d}{dt}(\overline{A}.\overline{B}) \qquad (ii)\ \dfrac{d}{dt}(\overline{A}\times\overline{B}) \qquad (iii)\ \dfrac{d}{dt}(\overline{A}.\overline{A})$.

Solution:

$$(i)\ \ (\overline{A}.\overline{B}) = (5t^2\ \overline{i} + t\ \overline{j} - t^3\ \overline{k}).(\sin t\ \overline{i} + \cos t\ \overline{j}) = 5t^2 \sin t + t\cos t$$

$$\therefore \frac{d(\overline{A}.\overline{B})}{dt} = \frac{d}{dt}(5t^2\sin t + t\cos t) = 5t^2\cos t + 10t\ \ \sin t + t(-\sin t) + \cos t$$

$$(ii)\ \overline{A}\times\overline{B} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 5t^2 & t & -t^3 \\ \sin t & \cos t & 0 \end{vmatrix} = t^3\cos t\ \overline{i} - t^3\sin t\ \overline{j} + (5t^2\cos t - t\sin t)\overline{k}$$

$$\therefore \frac{d}{dt}(\bar{A}\times\bar{B}) = \frac{d}{dt}[t^3\cos t\ \bar{i} - t^3\sin t\ \bar{j} + (5t^2\cos t - t\sin t)\bar{k}]$$

$$= (-t^3\sin t + 3t^2\cos t)\bar{i} - (t^3\cos t + 3t^2\sin t)\bar{j}$$

$$+ (-5t^2\sin t + 10\ t\cos t - t\cos t - \sin t)\bar{k}$$

$$(iii)\ \bar{A}.\bar{A} = (5t^2\ \bar{i} + t\ \bar{j} - t^3\ \bar{k}).(5t^2\ \bar{i} + t\ \bar{j} - t^3\ \bar{k}) = 25t^4 + t^2 + t^6$$

$$\therefore \frac{d}{dt}(\bar{A}.\bar{A}) = 100t^3 + 2t^2 + 6t^5$$

Example (3)

If $\bar{A} = (2x^2y - x^4)\bar{i} + (xy^3 - \sin x)\bar{j} + (x\cos y)\bar{k}$ Find

$$\frac{\partial\bar{A}}{\partial x},\quad \frac{\partial\bar{A}}{\partial y},\quad \frac{\partial^2\bar{A}}{\partial x^2},\quad \frac{\partial^2\bar{A}}{\partial y^2},\quad \frac{\partial^2\bar{A}}{\partial y\partial x},$$

Solution:

$$\frac{\partial\bar{A}}{\partial x} = \frac{\partial}{\partial x}[(2x^2y - x^4)\bar{i} + (xy^3 - \sin x)\bar{j} + (x\cos y)\bar{k}]$$

$$= \frac{\partial}{\partial x}(2x^2y - x^4)\bar{i} + \frac{\partial}{\partial x}(xy^3 - \sin x)\bar{j} + \frac{\partial}{\partial x}(x\cos y)\bar{k}$$

$$= (4xy - 4x^3)\bar{i} + (y^3 - \cos x)\bar{j} + (\cos y)\bar{k}$$

$$\frac{\partial\bar{A}}{\partial y} = \frac{\partial}{\partial y}[(2x^2y - x^4)\bar{i} + (xy^3 - \sin x)\bar{j} + (x\cos y)\bar{k}]$$

$$= \frac{\partial}{\partial y}(2x^2y - x^4)\bar{i} + \frac{\partial}{\partial y}(xy^3 - \sin x)\bar{j} + \frac{\partial}{\partial y}(x\cos y)\bar{k}$$

$$= (2x^2)\bar{i} + (3xy^2)\bar{j} + (-x\sin y)\bar{k}$$

$$\frac{\partial^2\bar{A}}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial}{\partial x}\bar{A}) = \frac{\partial}{\partial x}[(4xy - 4x^3)\bar{i} + (y^3 - \cos x)\bar{j} + (\cos y)\bar{k}]$$

$$= \frac{\partial}{\partial x}(4xy - 4x^3)\bar{i} + \frac{\partial}{\partial x}(y^3 - \cos x)\bar{j} + \frac{\partial}{\partial x}(\cos y)\bar{k} = (4y - 12x^2)\bar{i} + (\sin x)\bar{j}$$

$$\frac{\partial^2\bar{A}}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial}{\partial y}\bar{A}) = \frac{\partial}{\partial y}[(2x^2)\bar{i} + (3xy^2)\bar{j} + (-x\sin y)\bar{k}]$$

$$= \frac{\partial}{\partial y}(2x^2)\bar{i} + \frac{\partial}{\partial y}(3xy^2)\bar{j} + \frac{\partial}{\partial y}(-x\sin y)\bar{k} = (6xy\ )\bar{j} + (-x\cos y)\bar{k}$$

$$\frac{\partial^2 \bar{A}}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\bar{A}\right) = \frac{\partial}{\partial y}[(4xy - 4x^3)\bar{i} + (y^3 - \cos x)\bar{j} + (\cos y)\bar{k}]$$

$$= \frac{\partial}{\partial y}(4xy - 4x^3)\bar{i} + \frac{\partial}{\partial y}(y^3 - \cos x)\bar{j} + \frac{\partial}{\partial y}(\cos y)\bar{k} = (4x)\bar{i} + (3y^2)\bar{j} + (-\sin y)\bar{k}.$$

$$\frac{\partial^2 \bar{A}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\bar{A}\right) = \frac{\partial}{\partial x}[(2x^2)\bar{i} + (3xy^2)\bar{j} + (-x\sin y)\bar{k}]$$

$$= \frac{\partial}{\partial x}(2x^2)\bar{i} + \frac{\partial}{\partial x}(3xy^2)\bar{j} + \frac{\partial}{\partial x}(-x\sin y)\bar{k} = (4x)\bar{i} + (3y^2)\bar{j} + (-\sin y)\bar{k}.$$

The Differential Vector Operator $\bar{\nabla}$

The differential vector operator $\bar{\nabla}$ is a partial differential operator defined in

Cartesian by $\boxed{\bar{\nabla} = \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right)}$,

From this definition, one can find that

$$\bar{\nabla}.\bar{\nabla} = \bar{\nabla}^2 = \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right).\left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right)$$

$$\therefore \quad \bar{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Example (4):

Find the vector derivative to the scalar function $\varphi = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction $\bar{r} = 2\bar{i} - \bar{j} - 2\bar{k}$.

Solution

$$\bar{\nabla}\phi = \bar{\nabla}(x^2yz + 4xz^2) = \frac{\partial}{\partial x}(x^2yz + 4xz^2)\,\bar{i} + \frac{\partial}{\partial y}(x^2yz + 4xz^2)\,\bar{j} + \frac{\partial}{\partial z}(x^2yz + 4xz^2)\,\bar{k}$$

$$= (2xyz + 4z^2)\,\bar{i} + x^2z\,\bar{j} + (x^2y + 8xz)\,\bar{k} = 8\,\bar{i} - \bar{j} - 10\,\bar{k} \quad \text{at } (1,-2,-1)$$

The unit vector in the direction $\bar{r} = 2\bar{i} - \bar{j} - 2\bar{k}$ in the form

$$\bar{n} = \frac{\bar{r}}{|\vec{r}|} = \frac{2\bar{i} - \bar{j} - 2\bar{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\bar{i} - \bar{j} - 2\bar{k})$$

$\vec{\nabla}\varphi.\bar{n} = (8\vec{i} - \vec{j} - 10\vec{k}).(\frac{2}{3}\bar{i} - \frac{1}{3}\bar{j} - \frac{2}{3}\bar{k}) = \frac{37}{3}$. Thus vector derivative is

Example (5):

Find The unit vector in the direction normal to $x^2y + 2xz = 4$ at the point $(2,-2,3)$.

Solution:

Since $\bar{\nabla}\phi$ is a normal to the surface ϕ then the unit vector normal to ϕ in the form $\frac{\bar{\nabla}\phi}{|\bar{\nabla}\phi|}$. Now:

$$\bar{\nabla}\phi = \bar{\nabla}(x^2y + 2xz) = \frac{\partial}{\partial x}(x^2y + 2xz)\bar{i} + \frac{\partial}{\partial y}(x^2y + 2xz)\bar{j} + \frac{\partial}{\partial z}(x^2y + 2xz)\bar{k}$$

$$= (2xy + 2z)\,\bar{i} + x^2\,\bar{j} + (2x)\,\bar{k} = -2\,\bar{i} + 4\,\bar{j} + 4\,\bar{k} \quad \text{at } (2,-2,3).$$

$$\therefore \bar{n} = \frac{\bar{\nabla}\phi}{|\bar{\nabla}\phi|} = \frac{-2\,\bar{i} + 4\vec{j} + 4\bar{k}}{\sqrt{4+16+16}} = \frac{-1}{3}\bar{i} + \frac{2}{3}\bar{j} + \frac{2}{3}\bar{k}$$

Example (6):

Find $\vec{\nabla}\phi$ where: $(i)\ \varphi = \ln|\vec{r}|$ $(ii)\ \varphi = \frac{1}{|\vec{r}|}$ and $\vec{r}$ is the position vector for any point (x, y, z) in the surface φ.

Solution:

$$\because \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\therefore |\vec{r}| = \sqrt{x^2 + y^2 + z^2},$$

$$\ln|\vec{r}| = \frac{1}{2}\ln(x^2 + y^2 + z^2)$$

$$\therefore \vec{\nabla}\ln|\vec{r}| = \vec{\nabla}\frac{1}{2}\ln\left(x^2 + y^2 + z^2\right) = \frac{1}{2}\vec{\nabla}\ln\left(x^2 + y^2 + z^2\right)$$

$$=\frac{1}{2}\left\{\frac{\partial}{\partial x}\ln\left(x^2+y^2+z^2\right)\vec{i}+\frac{\partial}{\partial x}\ln\left(x^2+y^2+z^2\right)\vec{j}+\frac{\partial}{\partial x}\ln\left(x^2+y^2+z^2\right)\vec{k}\right\}$$

$$=\frac{1}{2}\left\{\frac{2x}{x^2+y^2+z^2}\vec{i}+\frac{2y}{x^2+y^2+z^2}\vec{j}+\frac{2z}{x^2+y^2+z^2}\vec{k}\right\}$$

$$=\frac{1}{x^2+y^2+z^2}(x\vec{i}+y\vec{j}+z\vec{k})=\frac{\vec{r}}{\left|\vec{r}\right|^2}.$$

$$\because \quad \frac{1}{\left|\vec{r}\right|}=\frac{1}{\sqrt{x^2+y^2+z^2}}=\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}$$

$$\therefore \quad \vec{\nabla}\frac{1}{\left|\vec{r}\right|}=\vec{\nabla}(x^2+y^2+z^2)^{\frac{-1}{2}}=-\frac{1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}}\vec{\nabla}(x^2+y^2+z^2)$$

$$=\frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}}[\frac{\partial}{\partial x}(x^2+y^2+z^2)\vec{i}+\frac{\partial}{\partial y}(x^2+y^2+z^2)\vec{j}+\frac{\partial}{\partial z}(x^2+y^2+z^2)\vec{k}]$$

$$\vec{\nabla}\frac{1}{\left|\vec{r}\right|}=\frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}}[2x\vec{i}+2y\vec{j}+2z\vec{k}]$$

$$=-(x^2+y^2+z^2)^{\frac{-3}{2}}(x\vec{i}+y\vec{j}+z\vec{k})=\frac{-(x\vec{i}+y\vec{j}+z\vec{k})}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

$$=\frac{-(x\vec{i}+y\vec{j}+z\vec{k})}{\left(\sqrt{x^2+y^2+z^2}\right)^3}=\frac{-\vec{r}}{\left|\vec{r}\right|^3}$$

Divergence of a Vector:

The divergence of a vector field (**div**) is a scalar field defined as the limit of the **flux** out of a small volume per unit volume as the volume tends to zero, the mathematical differential formula for ***div*** $\overline{A}$ where $\overline{A}=a_1\,i+a_2\,j+a_3\,k$ in the form

$$\boxed{\overline{\nabla}.\overline{A}=(\frac{\partial a_1}{\partial x}+\frac{\partial a_2}{\partial y}+\frac{\partial a_3}{\partial z})}$$

Example (7):

If $\vec{A} = x^2 z\vec{i} - 2y^3 z^2 \vec{j} + xy^2 z\vec{k}$ find $\vec{\nabla}.\vec{A}$.

Solution:

$$\vec{\nabla}.\vec{A} = div\,\vec{A} = (\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}).(x^2 z\vec{i} - 2y^3 z^2 \vec{j} + xy^2 z\ \vec{k})$$

$$= \frac{\partial}{\partial x} x^2 z + \frac{\partial}{\partial y}(-2y^3 z^2) + \frac{\partial}{\partial z} xy^2 z \quad = 2xz - 6y^2 z^2 + xy^2.$$

Example (8):

If $\phi = 2x^3 y^2 z^4$ find $\vec{\nabla}.\vec{\nabla}\phi$.

Solution:

$$\vec{\nabla}\phi = \vec{\nabla}(2x^3 y^2 z^4) = \frac{\partial}{\partial x}(2x^3 y^2 z^4)\vec{i} + \frac{\partial}{\partial y}(2x^3 y^2 z^4)\vec{j} + \frac{\partial}{\partial z}(2x^3 y^2 z^4)\vec{k}$$

$$= 6x^2 y^2 z^4 \vec{i} + 4x^3 yz^4 \vec{j} + 8x^3 y^2 z^3 \vec{k}$$

$$\vec{\nabla}.\vec{\nabla}\phi = (\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}).(6x^2 y^2 z^4 \vec{i} + 4x^3 y\ z^4\ \vec{j} + 8x^3 y^2 z^3\ \vec{k})$$

$$= \frac{\partial}{\partial x} 6x^2 y^2 z^4 + \frac{\partial}{\partial y} 4x^3 y\ z^4 + \frac{\partial}{\partial z} 8x^3 y^2 z^3 = 12xy^2 z^4 + 4x^3 z^4 + 24x^3 y^2 z^2.$$

Example (9):

find $\vec{\nabla}.\ \vec{\nabla}\frac{1}{|\vec{r}|}$ .where $\vec{r}$ is the position vector for any point (x, y, z) in the space.

Solution:

$$\vec{\nabla}\frac{1}{|\vec{r}|} = \vec{\nabla}(x^2 + y^2 + z^2)^{\frac{-1}{2}} = \frac{-1}{2}(x^2 + y^2 + z^2)^{\frac{-3}{2}}\ \vec{\nabla}(x^2 + y^2 + z^2)$$

$$= \frac{-1}{2}(x^2 + y^2 + z^2)^{\frac{-3}{2}}[\frac{\partial}{\partial x}(x^2 + y^2 + z^2)\vec{i} + \frac{\partial}{\partial y}(x^2 + y^2 + z^2)\vec{j} + \frac{\partial}{\partial z}(x^2 + y^2 + z^2)\vec{k}]$$

$$\vec{\nabla}\frac{1}{|\vec{r}|}=\frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}}[2x\vec{i}+2y\vec{j}+2z\vec{k}]$$

$$=-(x^2+y^2+z^2)^{\frac{-3}{2}}(x\vec{i}+y\vec{j}+z\vec{k})=\frac{-(x\vec{i}+y\vec{j}+z\vec{k})}{(x^2+y^2+z^2)^{\frac{3}{2}}}=\frac{-\vec{r}}{|\vec{r}|^3}$$

$$\therefore\ \vec{\nabla}.\left(\vec{\nabla}\frac{1}{|\vec{r}|}\right)=-\vec{\nabla}.\frac{\vec{r}}{|\vec{r}|^3}$$ Where $$\vec{\nabla}.(\phi\,\vec{A})=(\vec{\nabla}\phi).\vec{A}+\phi(\vec{\nabla}.\vec{A})$$

Let $\phi=\frac{1}{|\vec{r}|^3}$, $\vec{A}=\vec{r}$ then we have

$$\therefore \vec{\nabla}.\vec{\nabla}\frac{1}{|\vec{r}|}=\overline{-\nabla}.\frac{\vec{r}}{|\vec{r}|^3}=-\vec{\nabla}.(\frac{1}{|\vec{r}|^3}\vec{r})=-[(\vec{\nabla}\frac{1}{|\vec{r}|^3}).\vec{r})+\frac{1}{|\vec{r}|^3}(\vec{\nabla}.\vec{r})]$$

$$=-3|\vec{r}|^{-5}\vec{r}.\vec{r}-3|\vec{r}|^3=0$$

Curl of a Vector

The mathematical differential formula for ***curl*** $\bar{A}$ where $\bar{A}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$ is

$$\bar{\nabla}\times\bar{A}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ a_1 & a_2 & a_3\end{vmatrix}=(\frac{\partial a_3}{\partial y}-\frac{\partial a_2}{\partial z})\vec{i}+(\frac{\partial a_1}{\partial z}-\frac{\partial a_3}{\partial x})\vec{j}+(\frac{\partial a_2}{\partial x}-\frac{\partial a_1}{\partial y})\vec{k}$$

Example (10):

If $\vec{A}=x^2y\,\vec{i}-2xz\,\bar{j}+2yz\,\vec{k}$ Find $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$

Solution

$$(\vec{\nabla}\times\vec{A})=[(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k})\times(x^2y\vec{i}-2xz\bar{j}+2yz\vec{k})]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2y & -2xz & 2yz\end{vmatrix}$$

$$=[(\frac{\partial}{\partial y}(2yz)-\frac{\partial}{\partial z}(-2xz)]\vec{i}+[\frac{\partial}{\partial z}(x^2y)-\frac{\partial}{\partial x}(2yz)]\vec{j}+[\frac{\partial}{\partial x}(-2xz)-\frac{\partial}{\partial y}(x^2y)]\vec{k}=(2x+2z)\ \vec{i}-(x^2+2z)\ \vec{k}$$

$$\therefore \vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}\times[(2x+2z)\ \vec{i}-(x^2+2z)\ \vec{k}]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x+2z & 0 & -(x^2+2z)\end{vmatrix}=(2x+2)\vec{j}$$

Example (11):

If $\vec{F}=x^2yz\ \vec{i}+xyz\ \vec{j}-xyz^2\ \vec{k}$ Find $div\ \vec{F},\quad curl\ \vec{F}$

Solution:

$$div\ \vec{F}=\nabla\cdot\vec{F}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

$$=\frac{\partial}{\partial x}\left(x^2yz\right)+\frac{\partial}{\partial y}(xyz)+\frac{\partial}{\partial z}\left(-xyz^2\right)=2xyz+xz-2xyz=xz$$

$$curl\ \vec{F}=\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\vec{i}+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\vec{j}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\vec{k}$$

$$=\left[\frac{\partial}{\partial y}\left(-xyz^2\right)-\frac{\partial}{\partial z}(xyz)\right]\vec{i}+\left[\frac{\partial}{\partial z}\left(x^2yz\right)-\frac{\partial}{\partial x}\left(-xyz^2\right)\right]\vec{j}+\left[\frac{\partial}{\partial x}(xyz)-\frac{\partial}{\partial y}\left(x^2yz\right)\right]\vec{k}$$

$$=\left[\left(-xz^2\right)-(xy)\right]\vec{i}+\left[\left(x^2y\right)-\left(-yz^2\right)\right]\vec{j}+\left[(yz)-\left(x^2z\right)\right]\vec{k}$$

$$=\left(-xz^2-xy\right)\vec{i}+\left(x^2y+yz^2\right)\vec{j}+\left(yz-x^2z\right)\vec{k}$$

Example (12):

Show that $\vec{\nabla}\times\vec{\nabla}\phi=0$

Solution

$$\vec{\nabla}\times(\vec{\nabla}\phi)=\vec{\nabla}\times\left(\frac{\partial\phi}{\partial x}\vec{i}+\frac{\partial\phi}{\partial y}\vec{j}+\frac{\partial\phi}{\partial z}\vec{k}\right)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]\vec{i}+\left[\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)\right]\vec{j}+\left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right]\vec{k}$$

$$=\left[\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right]\vec{i}+\left[\frac{\partial^2\phi}{\partial z\partial x}-\frac{\partial^2\phi}{\partial x\partial z}\right]\vec{j}+\left[\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right]\vec{k}=0$$

Example (13):

Show that $\vec{\nabla}.(\vec{\nabla}\times\vec{A})=0$

Proof:

$$\text{let}\,\vec{A}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$$

$$(ii)\ \vec{\nabla}.(\vec{\nabla}\times\vec{A})=\vec{\nabla}.[(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k})\times(a_x\vec{i}+a_y\vec{j}+a_z\vec{k})]$$

$$\therefore\vec{\nabla}.(\vec{\nabla}\times\vec{A})=\vec{\nabla}.[(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z})\vec{i}+(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x})\vec{j}+(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y})\vec{k}]$$

$$=(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}).[(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z})\vec{i}+(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x})\vec{j}+(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y})\vec{k}]$$

$$=\frac{\partial}{\partial x}(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z})+\frac{\partial}{\partial y}(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x})+\frac{\partial}{\partial z}(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y})$$

$$=(\frac{\partial^2 a_z}{\partial x\partial y}-\frac{\partial a_y}{\partial x\partial z})+(\frac{\partial^2 a_x}{\partial y\partial z}-\frac{\partial^2 a_z}{\partial y\partial x})+(\frac{\partial^2 a_y}{\partial z\partial x}-\frac{\partial^2 a_x}{\partial z\partial y})=0$$

The Integrals of Vector Calculus

This is just a short guide to the many integrals we have defined, indicating how the computation of each can be reduced to computing single variable integrals. Since they can also be useful for computations, I have included the statements of Green’s, Stokes’ and Gauss’ theorems as well.

Line integral

Line integrals of Vector Fields:

If $\vec{F}(t) = F_x(t)\vec{i} + F_y(t)\vec{j} + F_z(t)\vec{k}$ and $r(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ Then the line integral of $\vec{F}(t)$ along the curve C is

$$\int_C \bar{F}.d\bar{r} = \int_C [F_x(t)\vec{i} + F_y(t)\vec{j} + F_z(t)\vec{k}].[dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}]$$

$$= \int_C F_x(t)dx + \int_C F_y(t)dy + \int_C F_z(t)dz$$

If C is closed curve then we denote to the integral by $\oint_C \bar{F}.d\bar{r}$

Physical meaning of line integral:

The line integral of the vector function $\vec{F}(x,y,z)$ along simple curve **C** from the point $P_1(x_1,y_1,z_1)$ to the point $P_2(x_2,y_2,z_2)$ represents to the work done by the force $\vec{F}(x,y,z)$ along the path joined to $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$.

Example (14):

Evaluate $\int_{p_1}^{p_2} \vec{F}.d\vec{r}$ for $\vec{F}(t) = (3x^2 + y)\vec{i} + 2y^3x\vec{j}$. From the point $P_1(0,0)$ to the point $P_2(1,1)$ in the following paths:

(i) $x = t, \quad y = t^2$.

(ii) along the path consisting of the straight lines from $(0,0)$ to $(1,0)$ and from $(1,0)$ to $(1,1)$.

(iii) the line segment to the points $(0,0)$ and $(1,1)$.

Solution:

$$\vec{F}.\,d\vec{r} = (3x^2 + y)dx + 2y^3xdy$$

$$\int_C \vec{F}(t).d\vec{r} = \int_{(0,0)}^{(1,1)} (3x^2+y)dx + 2y^3x\,dy \qquad \textbf{(1)}$$

from the equation of the path $x = t, \quad y = t^2$

$$\therefore dx = dt \qquad (2)$$
$$dy = 2t\,dt \qquad (3)$$

at the point $(0,0)$ $x = t = 0$ and at $(1,1)$ we find $x = t = 1$ then the new limit of the integral from 0 to 1.From (3),(2) in (1)we have:

$$\int_C \vec{F}.d\vec{r} = \int_{(0,0)}^{(1,1)} (3x^2+y)dx + 2y^3x\,dy$$

$$= \int_{t=0}^{1} (3t^2+t^2)dt + (2t^6)(t)(2tdt) = \int_{t=0}^{1} (4t^2+4t^8)dt = \frac{4t^3}{3} - \frac{4t^9}{9}\Big]_0^1 = \frac{4}{3} - \frac{4}{9} = \frac{8}{9}$$

(ii) From $(0,0)$ to $(1,0)$ and ending at $(1,1)$ the integral in the form:

$$\int_C \vec{F}.d\vec{r} = \int_{(0,0)}^{(1,0)} (3x^2+y)dx + 2y^3x\,dy + \int_{(1,0)}^{(1,1)} (3x^2+y)dx + 2y^3x\,dy$$

First integral $\displaystyle\int_{(0,0)}^{(1,0)} (3x^2+y)dx + 2y^3x\,dy$

on the line from $(0,0)$ to $(1,0)$ we have $y = 0$ then $dy = 0$ and x various from 0 to 1

Second integral $\displaystyle\int_{(1,0)}^{(1,1)} (3x^2+y)dx + 2y^3x\,dy$

on the line from $(1,0)$ to $(1,1)$ we have $x = 1$ then $dx = 0$ and y various from 0 to 1

$$\therefore \int_C \vec{F}.d\vec{r} = \int_{(0,0)}^{(1,0)} (3x^2+y)dx + 2y^3x\,dy + \int_{(1,0)}^{(1,1)} (3x^2+y)dx + 2y^3x\,dy$$

$$= \int_{x=0}^{1} (3x^2+0)dx + \int_{y=0}^{1} (0+2y^3)dy = \left.\frac{3x^3}{3}\right]_0^1 + \left.\frac{2y^4}{4}\right]_0^1 = 1+\frac{1}{2} = \frac{3}{2}$$

(iii) On the line from $(0,0)$ to $(1,1)$, its equation is $y = x$

at $(0,0) \Rightarrow y = x = 0$

at $(1,1) \Rightarrow y = x = 1$

$\because y = x \therefore dy = dx$

the integral in the form:

$$\int_C \vec{F}.d\vec{r} = \int_{(0,0)}^{(1,1)} (3x^2+y)dx + 2y^3x\,dy = \int_{x=0}^{1} (3x^2+x)dx + 2x^4dx$$

$$= \int_{x=0}^{1} (3x^2+x+2x^4)dx = \left.\frac{3x^3}{3}+\frac{x^2}{2}+\frac{2x^5}{5}\right]_0^1 = 1+\frac{1}{2}+\frac{2}{5} = \frac{19}{10}$$

$$\int_C \vec{F}.d\vec{r} = \int_{(0,0)}^{(1,1)} (3x^2+y)dx + 2y^3x\,dy = \int_{y=0}^{1} (3y^2+y)dy + 2y^4dy$$

$$= \int_{y=0}^{1} (3y^2+y+2y^4)dy = \left.\frac{3y^3}{3}+\frac{y^2}{2}+\frac{2y^5}{5}\right]_0^1 = 1+\frac{1}{2}+\frac{2}{5} = \frac{19}{10}$$

Example (15):

Find the work done W by the force $\vec{F} = 3xy\vec{i} - 5z\vec{j} + 10x\vec{k}$ from $t=0$ to $t=1$ along the path C which described by the equation $x = t^2+1, \quad y = 2t^2, \quad z = t^3$.

Solution:

the work done W by $\vec{F}$ given by $W = \int_C \vec{F}.\vec{dr}$

$$W = \int_C \vec{F}.d\vec{r} = \int_C (3xy\vec{i} - 5z\vec{j} + 10x\vec{k}).(dx\vec{i} + dy\vec{j} + dz\vec{k}) = \int_C 3xy\,dx - 5z\,dy + 10x\,dz$$

From path equation we have

$x = t^2+1 \quad \Rightarrow \quad dx = 2t\,dt, \quad y = 2t^2 \quad \Rightarrow \quad dy = 4t\,dt, \quad z = t^3 \quad \Rightarrow \quad dz = 3t^2dt$

$$\therefore W=\int_C \vec{F}\cdot d\vec{r}=\int_C 3xydx-5zdy+10xdz$$

$$=\int_0^1 3(t^2+1)(2t^2)(2t)dt-5t^3(4t)dt+10(t^2+1)(3t^2)dt$$

$$=\int_0^1 (12t^5+10t^4+12t^3+30t^2)dt=17$$

Example (16):

Evaluate $\int_C \vec{F}\cdot d\vec{r}$ form the point $(0,0)$ to the point $(1,2)$ along the curve C : $y=2x^2$ where $\vec{F}=3xy\vec{i}-y^2\vec{j}$.

Solution:

$\because \vec{r}=x\vec{i}+y\vec{j} \qquad \therefore d\vec{r}=dx\vec{i}+dy\vec{j}$

by using equation of the path we express the integral as a function of x (or y)by expressing the vector function $\vec{F}$ as a function of x (or y) only

$y=2x^2 \;\Rightarrow\; dy=4xdx \qquad x:0\rightarrow 1$

$$\therefore \int_C \vec{F}\cdot d\vec{r}=\int_C (3xy\vec{i}-y^2\vec{j})\cdot(dx\vec{i}+dy\vec{j})$$

$$=\int_C 3xy\,dx-y^2dy=\int_0^1 3x(2x^2)\,dx-(2x^2)^2(4x\,dx)$$

$$=\int_0^1 (6x^3-16x^5)dx=\frac{6}{4}x^4-\frac{16}{6}x^6\Big|_0^1=\frac{6}{4}-\frac{16}{6}=\frac{18-32}{12}=\frac{-14}{12}=\frac{-7}{6}$$

Note: We can solve the previous example by using parametric equation. Put $x=t$ Where t is a parameter ,at the point $(0,0)$ $x=t=0$, at the point $(1,2)$ $x=t=1$ and $y=2x^2=2t^2$.

Substitute in equation (1) by $x=t,\ dx=dt,\ y=2t^2\ dy=4tdt$ we get the same result.

Example (17):

Evaluate the work done by the force

$$\vec{F} = (2x - y + z)\vec{i} + (x + y - z^2)\vec{j} + (3x - 2y + 4z)\vec{k}$$

to moves a body along the circumference of the circle $x^2 + y^2 = 9$ one revolution.

Solution:

$z = 0$ In the plane of the circle then $\vec{F}$ takes the form

$$\vec{F} = (2x - y)\vec{i} + (x + y)\vec{j} + (3x - 2y)\vec{k}$$

the position vector for a point $P(x, y)$ on the circle is $\vec{r} = x\,\vec{i} + y\,\vec{j}$

$$\therefore d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

the work W done by the force $\vec{F}$ given as:

$$W = \int_C \vec{F}.d\vec{r} = \int_C [(2x - y)\vec{i} + (x + y)\vec{j} + (3x - 2y)\vec{k}].[dx\,\vec{i} + dy\,\vec{j} + 0\vec{k}]$$

$$= \int_C (2x - y)dx + (x + y)dy.$$

by using parametric equation of the circle

$$x = 3\cos\theta, \qquad y = 3\sin\theta$$

$$\therefore dx = -3\sin\theta\,d\theta, \quad dy = 3\cos\theta\,d\theta, \quad 0 \le \theta \le 2\pi$$

$$\therefore W = \int_C \vec{F}.d\vec{r} = \int_C (2x - y)dx + (x + y)dy$$

$$= \int_0^{2\pi} [2(3\cos\theta) - 3\sin\theta][-3\sin\theta]d\theta + [3\cos\theta + 3\sin\theta][3\cos\theta]d\theta$$

$$= \int_0^{2\pi} (9 - 9\sin\theta\cos\theta)d\theta = 9\theta - \frac{9}{2}\sin^2\theta\Big]_0^{2\pi} = 18\pi$$

Note: The path of integration in the previous example is closed. We donate to such this integral by $\oint_C \vec{F}.d\vec{r}$.

We consider the integration sign is positive when it is taken as anticlockwise and negative sign when it is taken as clockwise direction.

8. Independence pf the line integral on the path

For the vector function $\vec{F}$ If the value of the integral $\int_{p_1}^{p_2} \vec{F}.d\vec{r}$ independent to the path which joining the two points p_1 and p_2 then the vector field **F** is called a **conservative** vector field.

Theorem(1):

For the vector function $\vec{F}$ the value of the integral $\int_{p_1}^{p_2} \vec{F}.d\vec{r}$ independent on the path thought p_1, p_2 if and only if there exist a single valued scalar function ϕ such that $\vec{F} = \vec{\nabla}\phi$.

Theorem (2):

A vector field **F** is a **conservative** vector field if and only if $curl\ \vec{F} = \vec{\nabla} \times \vec{F} = 0$

Theorem (3):

$\int_{p_1}^{p_2} \vec{F}.d\vec{r}$ is independent of path in a domain **D** if and only if $\oint_C \vec{F}.d\vec{r} = 0$ for every closed path in **D**.

Definition(3):

A vector field **F** is called a **conservative** vector field if there is a scalar function φ such that $F = \nabla\varphi$. Here φ is called a **scalar potential function** of **F**.

Note: Line integrals of conservative vector fields are independent of path.

Theorem (4):

Let $F = P\ \bar{i} + Q\ \bar{j}$ be a conservative vector field, where all the partial derivatives are continuous. Then, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in D

Theorem (5):

Let $F = P\,\bar{i} + Q\,\bar{j}$ be a vector fields on an open simply-connected region **D**.

Suppose that all the partial derivatives are continuous and

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in D Then **F** is conservative.

Example (18):

Show that $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ is conservative field, find the scalar potential for this field and find the work done due to move body from $(1,-2,1)$ to $(3,1,4)$ in this field.

Solution:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} = 0$$

Then the field $\vec{F}$ is conservative and there exist scalar function φ such that $\vec{F} = \vec{\nabla}\varphi$ thus:

$$F.dr = \nabla\varphi.dr = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = d\varphi$$

$$\therefore d\varphi = F.dr = (2xy + z^3)dx + x^2dy + 3xz^2\,dz$$

$$= (2xydx + x^2dy) + (z^3dx + 3xz^2dz) = d(x^2y) + d(xz^3) = d(x^2y + xz^3)$$

$$\therefore \varphi = x^2y + xz^3 + c \qquad c \text{ is a constsnt}$$

and the work don in the form:

$$W = \int_{p_1}^{p_2} \vec{F}.d\vec{r} = \int_{p_1}^{p_2} (2xy + z^3)dx + x^2dy + 3xz^2dz = \int_{p_1}^{p_2} d\varphi$$

$$= x^2y + xz^3\Big|_{p_1}^{p_2} = x^2y + xz^3\Big|_{(1,-2,1)}^{(3,1,4)} = 202$$

9 - Surface Integrals

9.1 Surface Integrals of Vector Fields:

If $\vec{F}$ is a continuous vector field on an oriented surface **S** with unit normal vector $\vec{n}$. Then, the Surface Integrals of $\vec{F}$ over **S** is:

$$\iint_S \vec{F}.\vec{n}\, dS \qquad (*)$$

The integral is also called the flux across **S.**

And $dS = \dfrac{dx\,dy}{|\vec{k}.\vec{n}|}$ then

$$\boxed{\iint_S \vec{F}.\vec{n}\,dS = \iint_A \vec{F}.\vec{n}\,\frac{dx\,dy}{|\vec{k}.\vec{n}|}}$$

Example (19):

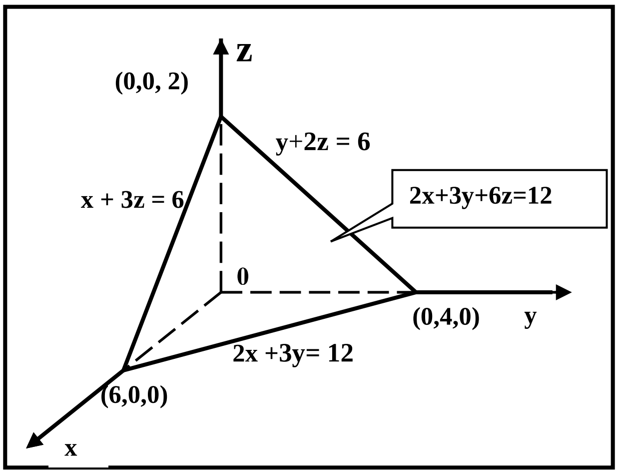
Evaluate $\iint_S \vec{F}.\vec{n}\,dS$ where $\vec{F} = 18\vec{i} - 12\vec{j} + 3\vec{k}$ and S is the plane

$2x + 3y + 6z = 12, \quad x > 0,\ y > 0,\ z > 0.$

Solution:

$$\iint_S \vec{F}.\vec{n}\,dS = \iint_A \vec{F}.\vec{n}\,\frac{dx\,dy}{|\vec{k}.\vec{n}|}$$

$$\vec{n} = \frac{\vec{\nabla}\varphi}{|\vec{\nabla}\varphi|} = \frac{\vec{\nabla}(2x+3y+6z)}{|\vec{\nabla}(2x+3y+6z)|} = \frac{1}{7}(2\vec{i} + 3\vec{j} + 6\vec{k})$$



$$\therefore \vec{k}.\vec{n} = \frac{6}{7}, \quad dS = \frac{dxdy}{|\vec{k}.\vec{n}|} = \frac{7\,dxdy}{6}$$

$$\vec{F}.\vec{n} = (18\vec{i} - 12\vec{j} + 3\vec{k}).\frac{1}{7}(2\vec{i} + 3\vec{j} + 6\vec{k}) = \frac{18}{7}$$

$$\vec{F}.\vec{n}\,dS = \vec{F}.\vec{n}\,\frac{dx\,dy}{|\vec{k}.\vec{n}|} = (\frac{18}{7})\,\frac{7dxdy}{6} = 3\,dxdy$$

$$\therefore \iint_S \vec{F}.\vec{n}\,dS = \iint_A \vec{F}.\vec{n}\,\frac{dxdy}{|\vec{k}.\vec{n}|} = \iint_A 3dxdy = \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} 3dxdy = \int_{x=0}^{6} (12-2x)dx$$

$$= 12x - x^2\Big|_0^6 = 72 - 36 = 36$$

10 - Volume integral:

If S is a surface enclosed by the volume V then we define the Volume integral of the scalar function $\varphi(x,y,z)$ over the volume V as:

$$\iiint_V \varphi(x,y,z)dV = \iiint_V \varphi(x,y,z)dxdydz$$

and volume integral of the vector function $\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ over the volume V as:

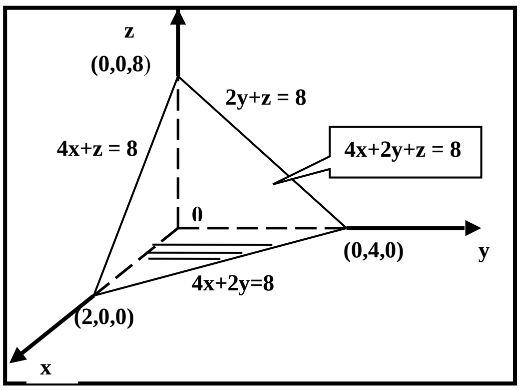
$$\iiint_V \vec{F}dV = \iiint_V (F_x\vec{i} + F_y\vec{j} + F_z\vec{k})\,dxdydz$$

$$= \vec{i}\iiint_V F_x\,dxdydz + \vec{j}\iiint_V F_y\,dxdydz + \vec{k}\iiint_V F_z\,dxdydz$$

Example (20):

Evaluate $\iiint_V \varphi(x,y,z)dV$ where $\varphi(x,y,z) = 45x^2y$ over the region bounded by the planes: $x=0,\ y=0,\ z=0,\ 4x+2y+z=8$

Solution:



the boundary of the volume can be determined as following:

To find the intersection of the volume with x-axis put $y=z=0$ in the equation $4x+2y+z=8$ to get the point $(2,0,0)$.Similarly with

y -axis(put $x=z=0$) and z -axis ($x=y=0$) to get $(0,4,0),(0,0,8)$ respectively.

The boundary of the volume can be determined as following:

To find the intersection of the volume with xy -plane put $z=0$ in the equation $4x+2y+z=8$ to get the straight line $2x+y=4$.

Similarly with yz -plane(put $x=0$) and xz -plane ($y=0$) see the figure.

$x:0\to 2, y:0\to 4-2x$ and $z:0\to 8-4x-2y$

$$\iiint_V \varphi(x,y,z)dV = \int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y}(45x^2y)dzdydx = 45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y\,[z]_0^{8-4x-2y}\,dxdy$$

$$=45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y\,(8-4x-2y)dxdy = 45\int_{x=0}^{2}\int_{y=0}^{4-2x}(8x^2y-4x^3y-2x^2y^2)dxdy$$

$$\iiint_V \varphi(x,y,z)dV = \int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y}(45x^2y)dzdydx = 45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y\,[z]_0^{8-4x-2y}\,dxdy$$

$$=45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y\,(8-4x-2y)dxdy = 45\int_{x=0}^{2}\int_{y=0}^{4-2x}(8x^2y-4x^3y-2x^2y^2)dxdy$$

$$=45\int_{x=0}^{2}[(8x^2-4x^3)\frac{y^2}{2}-\frac{2y^3}{3}]\Big|_0^{4-2x}dx$$

$$=45\int_{x=0}^{2}[(8x^2-4x^3)\frac{(4-2x)^2}{2}-\frac{2(4-2x)^3}{3}]dx = 128$$

11 - Integral Theorems

11.1- Green's Theorem

Green’s theorem relates line integrals in the plane to double integrals.

Theorem(6):

If **C** be a positively oriented, piecewise-smooth, simple closed curve in the plane and let **R** be the region bounded by **C.** Suppose $F(x,y)=M(x,y)\vec{i}+N(x,Y)\vec{j}$ is a vector field such that all the partial derivatives of **M** and **N** are continuous on an open region contains **R**, then

$$\oint_C M(x,y)dx + N(x,y)dy = \iint_R \left(\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right)dxdy$$

Application:

The formulas to find the area of R : $\oint_C xdy - ydx$

applying Green's Theorem on the integral $\oint_C xdy - ydx$

$$M = -y, \quad N = x \qquad \Rightarrow \qquad \frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1$$

$$\therefore \oint_C M(x,y)dx + N(x,y)dy = \oint_C (-y)dx + xdy$$

$$= \oint_C xdy - ydx = \iint_R \left(\frac{\partial (x)}{\partial x} - \frac{\partial (-y)}{\partial y}\right)dxdy = \iint_R 2dxdy$$

and $\iint_R dxdy = \frac{1}{2}\oint_C xdy - ydx$

Example (21):

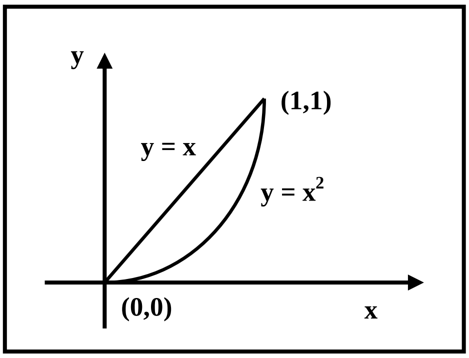
Verify Green's theorem for the integral $\oint_C (xy + y^2)dx + x^2 dy$

Where **C** is the closed path consisting of $y = x^2, \quad y = x$.

Solution:

Calculation of the Line integral $\oint_C M(x,y)dx + N(x,y)dy$

the point of intersection of the two curves $y = x^2, \quad y = x$ are $(0,0),(1,1)$ see the figure.



$$\therefore I=\oint_C (xy+y^2)dx+x^2dy$$

$$=\int_{\substack{(0,0)\\ on\ y=x^2}}^{(1,1)} (xy+y^2)dx+x^2dy+\int_{\substack{(1,1)\\ on\ y=x}}^{(0,0)} (xy+y^2)dx+x^2dy \qquad (1)$$

On $y=x^2, \quad dy=2x\,dx$

$$\therefore \int_{\substack{(0,0)\\ on\ y=x^2}}^{(1,1)} (xy+y^2)dx+x^2dy$$

$$=\int_0^1 (x(x^2)+x^4)dx+x^2(2x)dx=\int_0^1 (3x^3+x^4)\,dx=\frac{19}{20} \quad ....(2)$$

on $y=x \Rightarrow dx=dy$

$$\therefore \int_{\substack{(1,1)\\ on\ y=x}}^{(0,0)} (xy+y^2)dx+x^2dy=\int_1^0 (x(x)+x^2)dx+x^2dx=\int_1^0 3x^2dx=-1 \quad ....(3)$$

from (2),(3) in (1)

$$I=\oint_C (xy+y^2)dx+x^2dy=\frac{19}{20}-1=-\frac{1}{20} \quad ....(4)$$

Calculation of the double integral:

$$\iint_S \left(\frac{\partial N(x,y)}{\partial x}-\frac{\partial M(x,y)}{\partial y}\right)dx\,dy$$

we have $M(x,y)=xy+y^2, \quad N(x,y)=x^2$ then

$$\iint_S \left(\frac{\partial N(x,y)}{\partial x}-\frac{\partial M(x,y)}{\partial y}\right)dxdy=\iint_R \left[\frac{\partial}{\partial x}(x^2)-\frac{\partial}{\partial y}(xy+y^2)\right]dxdy$$

$$=\int_{x=0}^{1}\int_{y=x^2}^{x} (x-2y)dydx=\int_{x=0}^{1} (xy-y^2)\Big|_{x^2}^{x} dx=\int_{x=0}^{1} (x^4-x^3)dx=-\frac{1}{20} \quad ....(5)$$

From (4),(5) we find

$$\oint_C M(x,y)dx + N(x,y)dy = \iint_S \left(\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right)dxdy = \frac{1}{20}$$

Example (22):

Evaluate $\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2\,dy$ along the path $x^4 - 6xy^3 = 4y^2$.

Solution:

We note that the path is implicit function then if the integral independent on the path we chose another path for the integration.

$M(x,y) = 10x^4 - 2xy^3, \quad N(x,y) = -3x^2y^2$

$\frac{\partial M}{\partial y} = -6y^2x \qquad \frac{\partial N}{\partial x} = -6y^2x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -6xy$

then the integral independent on the path

we take the straight line which joint to the points $(0,0),(2,1)$ as a new path for the integration the equation of this path is

$y = \frac{1}{2}x$ then $dy = \frac{1}{2}dx$ and

$$\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2\,dy$$

$$= \int_0^2 \left(10x^4 - 2x\left(\frac{1}{2}x\right)^3\right)dx - 3x^2\left(\frac{1}{2}x\right)^2\left(\frac{1}{2}dx\right) = \int_0^2 \left(\frac{75x^4}{80}\right)dx = 60$$

Example (23):

Find the area enclosed by $x = a\cos\theta, \quad y = b\sin\theta$.

Solution:

The area **A** inside closed curve $x = x(t), y = y(t)$ determined by

$$\iint_R dxdy = \frac{1}{2}\oint_C xdy - ydx$$

$\because x = a\cos\theta, \quad y = b\sin\theta \qquad \therefore dx = -a\sin\theta d\theta, \quad dy = b\cos\theta d\theta$ and

$$A = \frac{1}{2}\oint_C xdy - ydx = \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(b\cos\theta)d\theta - (b\sin\theta)(-a\sin\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta)d\theta = ab\int_0^{2\pi} d\theta = \pi ab.$$

11.2 - Divergence Theorem

The **divergence theorem**, also known as **Gauss' theorem**, relates flux integrals over closed surfaces to triple integrals over the regions that they contain. Specifically,

Let **V** be a simple solid region and let **S** be the boundary surface of **V** with positive (outward) orientation. Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be a vector field whose component Functions has continuous partial derivatives on an open region $\supseteq V$ Then

$$\iiint_V \vec{\nabla}.\vec{F}\ dV = \iint_S \vec{F}.\vec{n}\ ds$$

where $\vec{n}$ is the unit vector to the surface **S**

Example (24):

Evaluate $\iint_S \vec{F}.\ \vec{n}\ ds$ where $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface the cube bounded by $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$

Solution:

By applying divergence theorem

$$\vec{\nabla}.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial(4xz)}{\partial x} + \frac{\partial(-y^2)}{\partial y} + \frac{\partial(yz)}{\partial z} = 4x - y$$

$$\iint_S \vec{F}.\vec{n}\,ds = \iiint_V \vec{\nabla}.\vec{F}\,dV = \iiint_V (4x-y)\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 (4x-y)\,dx\,dy\,dz = \frac{3}{2}$$

Example (25):

verify **divergence theorem** for the vector field $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + +z^2\vec{k}$ on the surface of the cylinder $x^2 + y^2 = 4,\ z = 0,\ z = 3$

Solution:

determination of the volume integral: I_v -

$$\because \vec{\nabla}.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial(4x)}{\partial x} + \frac{\partial(-2y^2)}{\partial y} + \frac{\partial(z^2)}{\partial z} = 4 - 4y + 2z$$

$$\therefore I_v = \iiint_V \vec{\nabla}.\vec{F}\,dV = \iint_R \int_{z=0}^{3} (4-4y+2z)\,dz\,dx\,dy = \iint_R (4z - 4yz + z^2)\Big|_{z=0}^{z=3} dx\,dy = \iint_R (21-12y)\,dx\,dy$$

using polar co-ordinate to calculate $\iint_R (21-12y)\,dx\,dy$

where **R** is the circler $x^2 + y^2 = 4$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad dx\,dy = r\,dr\,d\theta$$

$$\therefore \iiint_V \vec{\nabla}.\vec{F}\,dV = \iint_R (21-12y)\,dx\,dy = \int_0^2\int_0^{2\pi} (21 - 12r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^2 (21r\theta + 12r^2\cos\theta)\Big|_{\theta=0}^{2\pi} dr$$

$$= \int_0^2 (42\pi r)\,dr = 21\pi r^2\Big|_0^2 = 84\pi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

determination of the surface integral: I_S -

the closed surface consists of

S_1 with equation $z = 0$ (lower base)

S_2 with equation $z = 3$ (upper base)

S_3 with equation $x^2 + y^2 = 4$

$$\therefore I_s = \iint_S \vec{F}.\vec{n}ds = \iint_{S_1} \vec{F}.\vec{n}ds + \iint_{S_2} \vec{F}.\vec{n}ds + \iint_{S_3} \vec{F}.\vec{n}ds \text{ ......(2)}$$

On S_1

$$\vec{n} = -\vec{k}, \qquad z = 0,$$

$$\therefore \vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k} = 4x\vec{i} - 2y^2\vec{j}$$

$$\vec{F}.\vec{n} = 0 \qquad \Rightarrow \iint_{S_1} \vec{F}.\vec{n}ds = 0 \text{ ....(3)}$$

On S_2

$$\vec{n} = \vec{k}, \qquad z = 3, \qquad \therefore \vec{F} = 4x\vec{i} - 2y^2\vec{j} + 9\vec{k}$$

$$\vec{F}.\vec{n} = 9 \qquad \Rightarrow \iint_{S_2} \vec{F}.\vec{n}ds = 9\iint_{S_2} ds = 9(4\pi) = 36\pi \text{ ....(4)}$$

On S_3

$$\vec{n} = \frac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|} = \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{4x^2 + 4y^2}} = \frac{1}{2}(x\vec{i} + y\vec{j})$$

Parametric equation to S_3

$$x = 2\cos\theta, \; y = 2\sin\theta$$

$$\vec{F}.\vec{n} = (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}).\frac{1}{2}(x\vec{i} + y\vec{j}) = 2x^2 - y^3$$

$$\therefore \iint_{S_3} \vec{F}.\vec{n}ds = \iint_{S_3} (2x^2 - y^3)ds$$

$$ds = 2dz\,d\theta$$

$$\therefore \iint_{S_3} \vec{F}.\vec{n}ds = \int_{\theta=0}^{2\pi}\int_{z=0}^{3} [2(2\cos\theta)^2 - (2\sin\theta)^3]2dz\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{z=0}^{3} (16\cos^2\theta - 16\sin^3\theta)dzd\theta$$

$$= 3\int_{\theta=0}^{2\pi} (16\cos^2\theta - 16\sin^3\theta)d\theta = 48\pi \text{ ....(5)}$$

by substituting from (3),(4) and (5) in (2) we get

$$I_s = \iint_S \vec{F}.\vec{n}ds = 0 + 36\pi + 48\pi = 84\pi \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

$$\therefore I_s = I_v .$$

<u>We use the following integrals</u>

$$\int_0^{2\pi} \cos^2\theta d\theta = \int_0^{2\pi} \frac{1}{2}(1+\cos 2\theta)d\theta = \frac{1}{2}(\theta + \frac{1}{2}\sin\theta) \Big|_0^{2\pi} = \pi$$

$$\int_0^{2\pi} \sin^3\theta d\theta = \int_0^{2\pi} \sin^2\theta \sin\theta d\theta = \int_0^{2\pi} (1-\cos^2\theta)\sin\theta d\theta$$

$$= \int_0^{2\pi} \sin\theta d\theta - \int_0^{2\pi} \cos^2\theta \sin\theta d\theta = \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{2\pi} = 0$$

11.3 - <u>Stoke's Theorem:</u>

Stokes' theorem relates line integrals around curves in $\mathbb{R}^3$ so certain flux integrals.

If $\vec{F}$ vector field and **S** is an oriented surface with boundary **C** given the induced orientation, then

$$\oint_C \vec{F}.d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}).\vec{n}dS$$

Green's theorem is the special case of Stokes' theorem in which the surface **S** is a subset of the plane.

<u>Example (26):</u>

Verify **Stokes' theorem** for the vector field function $\vec{F} = (2x - y)\vec{i} - yz^2\vec{j} - y^2 z\vec{k}$ over the surface of the sphere $x^2 + y^2 + z^2 = 1, \quad z \geq 0$.

<u>Solution :</u>

The bound of the surface at $z = 0$ is the curve $x^2 + y^2 = 1$ Then $\vec{F} = (2x - y)\vec{i}$ and by using parametric equations $x = \cos t, \quad y = \sin t$ then $dx = -\sin t dt$

The line integral:

$$\oint_C \vec{F}.d\vec{r} = \oint_C (2x - y)dx = \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)dt$$

$$= \int_0^{2\pi} (-2\cos t \sin t + \sin^2 t)dt = \int_0^{2\pi} \left(-2\cos t \sin t + \frac{1}{2}(1 - \cos 2t)\right)dt$$

$$= \cos^2 t + \frac{1}{2}t - \frac{1}{4}\sin 2t \Big|_0^{2\pi} = \pi \quad ....(22)$$

The surface integral:

First we find $\vec{\nabla} \times \overline{F}$ as following

$$\vec{\nabla} \times \overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \vec{k}$$

$$\therefore \iint_S (\vec{\nabla} \times F).\vec{n}\, dS = \iint_S \vec{k}.\vec{n}\, ds = \iint_R dx\, dy = \pi$$

where $\vec{k}.\vec{n}ds = dxdy$ and the value of the integral $\iint_R dxdy$ is the area of the circle

$x^2 + y^2 = 1$ which is equal π

another method to find $\iint_R dxdy$ is using polar co-ordinates then

$$\iint_R dx\, dy = \int_0^1 \int_0^{2\pi} r\, dr\, d\theta = 2\pi(\frac{r^2}{2}) \Big|_0^1 = \pi \quad ....(23)$$

From (22),(23) **Stokes' theorem is** verified**.**

Fourier series

Let us assume that $f(x)$ is a periodic function of period 2π which can represented by the trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \qquad (1)$$

Given such a function $f(x)$, we want to determine the coefficients a_n and b_n in the corresponding series

$$\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx} \qquad (2)$$

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx} \qquad (3)$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx} \qquad (4)$$

Example(1):

Find the Fourier series of the periodic function $f(x)$ where

$$f(x) = \begin{cases} -k & when \quad -\pi < x < 0 \\ k & when \quad 0 < x < \pi \end{cases}$$

and $f(x) = f(x + 2\pi)$

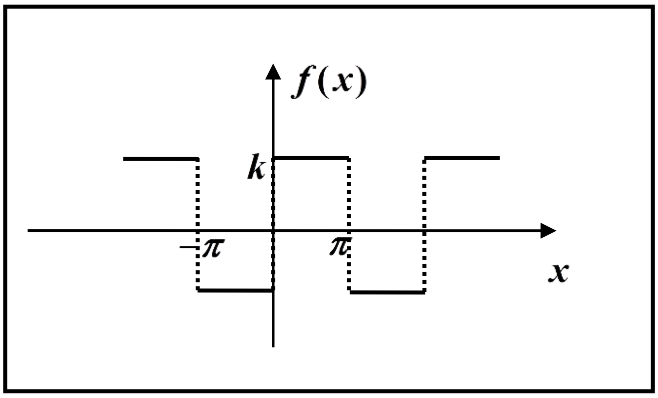
Hence find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)}$.

Solution:

the graph of the given function as a periodic function with period 2π shown in the following

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -kdx + \int_{0}^{\pi} kdx\right]$$

$$= \frac{k}{2\pi}\left[\int_{-\pi}^{0} -dx + \int_{0}^{\pi} dx\right] = \frac{k}{2\pi}\left[-x \Big|_{-\pi}^{0} + x \Big|_{0}^{\pi}\right] = \frac{k}{2\pi}[-\pi + \pi] = 0 \qquad (i)$$



From (3)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-k)\cos nx\,dx + \int_{0}^{\pi} (k)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-k}{n}\sin nx \Big|_{-\pi}^{0} + \frac{k}{n}\sin nx \Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-k}{n}\{\sin 0 - \sin(-n\pi)\} + \frac{k}{n}\{\sin(n\pi) - \sin 0\}\right] = 0 \qquad (ii)$$

and from (4)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-k)\sin nx\,dx + \int_{0}^{\pi} (k)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{k}{n}\cos nx \Big|_{-\pi}^{0} - \frac{k}{n}\cos nx \Big|_{0}^{\pi}\right]$$

$$= \frac{k}{\pi n}\left[\{\cos 0 - \cos(n\pi)\} - \{\cos(n\pi) - \cos 0\}\right]$$

$$= \frac{2k}{\pi n}\left[\cos 0 - \cos(n\pi)\right] = \frac{2k}{\pi n}\left[1 - (-1)^n\right]$$

$$b_n = \frac{2k}{\pi n}\left[1 - (-1)^n\right] = \begin{cases} 0 & \text{for } n \text{ is even} \\ \dfrac{4k}{\pi n} & \text{for } n \text{ is odd} \end{cases}$$

Hence $b_{2n-1} = \dfrac{4k}{(2n-1)\pi}, \; n = 1, 2, 3, \ldots \qquad (iii)$

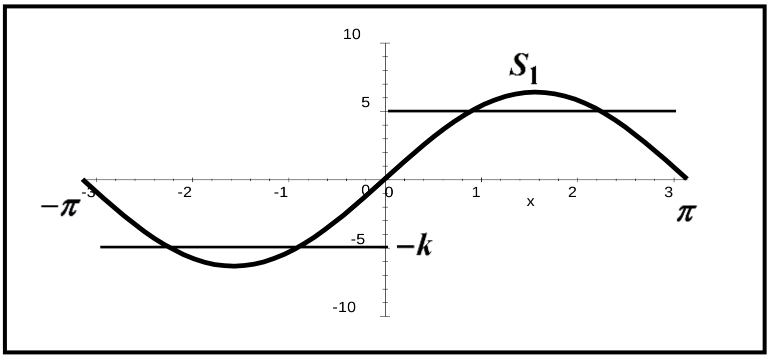
$$b_1 = \frac{4k}{\pi},\ b_3 = \frac{4k}{3\pi},\ b_5 = \frac{4k}{5\pi},\ldots$$

Substitute from (**i**),(**ii**) and (**iii**)in (1)

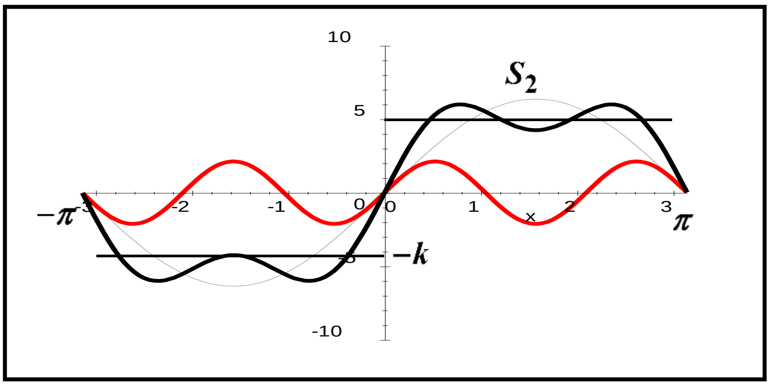
$$f(x) = \sum_{n=1}^{\infty} \frac{4k}{(2n-1)\pi} \sin(2n-1)x$$

and the partial sums are

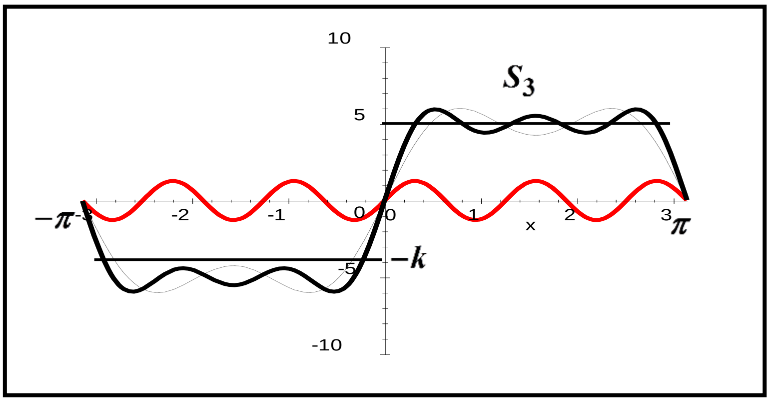
$$S_1 = \frac{4k}{\pi}\sin x,$$



$$S_2 = \frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x\right],$$



$$S_3 = \frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right]$$



Assuming that $f(x)$ is the sum of the series and setting $x=\frac{\pi}{2}$ we have

$$f\left(\frac{\pi}{2}\right)=k=\sum_{n=1}^{\infty}\frac{4k}{(2n-1)\pi}\sin(2n-1)\frac{\pi}{2}=\sum_{n=1}^{\infty}\frac{4k(-1)^{n-1}}{(2n-1)\pi}$$

$$\therefore \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)}=\frac{\pi}{4}$$

5. Even and odd functions:

6.1. Fourier cosine series

The Fourier series of an even function $f(t)$ having period $2T$ is a **Fourier cosine series** in the form.

$$f(t)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{T}t \qquad (10)$$

With coefficients

$$a_0=\frac{1}{T}\int_0^T f(t)dt \qquad (11)$$

$$a_n=\frac{2}{T}\int_0^T f(t)\cos\frac{n\pi t}{T}dt,\ n=1,2,3,\ldots \qquad (12)$$

Since the function $f(t)$ is even then $f(t)\sin\frac{n\pi t}{T}$ is an odd function hence

$$b_n=\frac{2}{T}\int_0^T f(t)\sin\frac{n\pi t}{T}dt=0,\quad \forall n=1,2,3,\ldots$$

6.2. Fourier sin series

The Fourier series of an odd function $f(t)$ having period $2T$ is a **Fourier sine series** in the form.

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{T} t \qquad (13)$$

With coefficients

$$\boxed{b_n = \frac{2}{T}\int_0^T f(t)\sin\frac{n\pi t}{T}\,dt,\ n = 1,2,3,\ldots} \qquad (14)$$

Since the function $f(t)$ is odd then $a_0 = \frac{1}{T}\int_0^T f(t)dt = 0$ and $f(t)\cos\frac{n\pi t}{T}$ is an odd function hence

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{n\pi t}{T} dt = 0, \quad \forall n = 1,2,3,\ldots$$

Example(4):

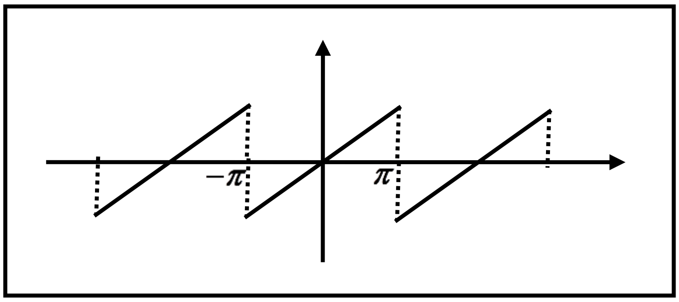
Find the Fourier series of the function

$$f(x) = x \quad when\ -\pi < x < \pi \ \ and \ \ f(x) = f(x+2\pi).$$

Solution:

Since the function is an odd then the it be represented as Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

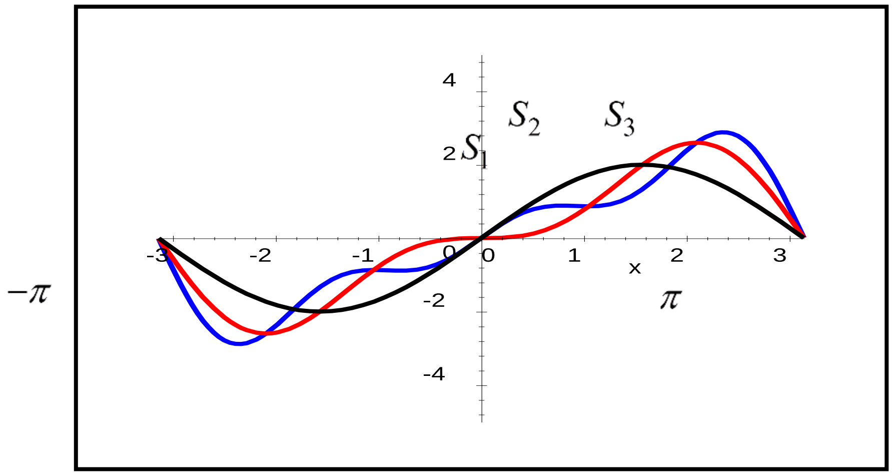


$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] = \frac{2}{n}(-1)^{n+1}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

$$= 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right]$$



Complex Variable

Example (1):

Show that the function $\mathbf{f(x) = z^3}$ satisfy Cauchy Riemann Equations hence deduce

$f'(z)$

Solution:

We note that

$$f(z) = z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$$\therefore u(x,y) = x^3 - 3xy^2, \qquad v(x,y) = 3x^2y - y^3$$

$$\therefore u_x = 3x^2 - 3y^2, \qquad v_x = 6xy$$

$$u_y = -6xy, \qquad vy = 3x^2 - 3y^2$$

$$\therefore u_x = v_y, \qquad u_y = -v_x$$

Then Cauchy Riemann equations are satisfied and

$$f'(z) = u_x + iv_x = 3x^2 - 3y^2 + i(6xy)$$
$$= 3[(x^2 - y^2) + i(2xy)] = 3(x+iy)^2 = 3z^2$$

Example (2):

Show that the function $u = e^x \sin y$ is harmonic function and find the function v such that $f = u + iv$ satisfy Cauchy-Riemann equations.

Solution:

$$u = e^x \sin y \qquad \frac{\partial u}{\partial x} = e^x \sin y, \quad \frac{\partial^2 u}{\partial x^2} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = e^x \cos y, \quad \frac{\partial^2 u}{\partial y^2} = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \sin y - e^x \sin y = 0$$

so that u is harmonic now we obtain the function v such that u and v satisfies the equations

$$(i)\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (ii)\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

integrate **(i) with respect to** y we have

$$v = \int \frac{\partial u}{\partial x}\, dy + f(x) = \int e^x \sin y\, dy + f(x) = -e^x \cos y + f(x)$$

where $f(x)$ is the integration constant

to determine $f(x)$ we use (ii) as follows

$$\frac{\partial v}{\partial x} = -e^x \cos y + f'(x) = -\frac{\partial u}{\partial y} = -e^x \cos y$$

$\therefore f' = 0 \Rightarrow f(x) = C$ (pure arbitrary constant)

Hence $v = -e^x \cos y + C$ and

$$f(z) = u + iv = e^x \sin y - ie^x \cos y + C = -ie^x(\cos y + i \sin y) + C = -ie^z + C$$

where C is an arbitrary constant.

Example (3):

Show that $\overline{\sin z} = \sin \bar{z}$.

Solution:

$$\overline{\sin z} = \overline{\sin(x+iy)}$$
$$= \overline{\sin x \cos(iy) + \cos x \sin(iy)} = \overline{\sin x \cosh y + i \cos x \sinh y}$$
$$= \sin x \cosh y - i \cos x \sinh y = \sin x \cos(iy) - \cos x \sin(iy)$$
$$= \sin(x - iy) = \sin \bar{z} \qquad \text{then } \overline{\sin z} = \sin \bar{z}$$

Example (4):

Evaluate $\int_C \bar{z}^{-2}\, dz$ in the following cases

(i) C is the circle $|z| = 1$.

(ii) C is the circle $|z-1| = 1$.

Solution:

(i) the parametric equation is

$$z = e^{it} \Rightarrow dz = ie^{it}dt \quad 0 \le t \le 2\pi \quad and \quad \bar{z}^{-2} = \left(e^{-it}\right)^{2} = e^{-2it}$$

$$\therefore \int_C \bar{z}^{-2} dz = \int_0^{2\pi} e^{-2it}.ie^{it}dt = i\int_0^{2\pi} e^{-it}dt = i\left[\frac{e^{-it}}{-i}\right]_0^{2\pi} = -\left[e^{-2\pi i} - 1\right] = 0$$

(ii) by using the parametric equation of the circle $|z-1| = 1$ we find

$$z - 1 = e^{it} \Rightarrow z = 1 + e^{it} \Rightarrow dz = ie^{it}\,dt, \quad 0 \le t \le 2\pi$$
$$\bar{z} = \overline{1 + e^{it}} = 1 + \overline{e^{it}} = 1 + e^{-it}$$
$$\bar{z}^{-2} = \left(1 + e^{-it}\right)^{2} = 1 + 2e^{-it} + e^{-2it}$$
$$\therefore \int_C \bar{z}^{-2} dz = \int_0^{2\pi} (1 + 2e^{-it} + e^{-2it}).(ie^{it}\,dt) = i\int_0^{2\pi} (e^{it} + 2 + e^{-it})dt$$
$$= i\left[\frac{e^{it}}{i} + 2t + \frac{e^{-it}}{-i}\right]_0^{2\pi} = i\left[\frac{e^{2\pi i} - 1}{i} + 4\pi + \frac{e^{-2\pi i} - 1}{-i}\right] = 4\pi i$$

Example (5):

Evaluate $\oint_C \frac{e^z}{z-2}\,dz$ for any closed contour contains the point $z_0 = 2$.

Solution:

$\frac{f(z)}{z-z_0} = \frac{e^z}{z-2}$ then $f(z) = e^z,\ z_0 = 2$ *and* $f(z_0) = e^2$

apply Cauchy's integral Formula $\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\ f(z_0)$

then $\oint_C \frac{e^z}{z-2}\,dz = 2\pi i\ f(2) = 2\pi i\, e^2$

Example (6):

Evaluate $\oint_C \frac{z^3-6}{2z-i}\,dz$ where C is the circle $|z| = 2$

Solution:

$$\oint_C \frac{z^3-6}{2z-i}\,dz = \frac{1}{2}\oint_C \frac{z^3-6}{z-\frac{1}{2}i}\,dz$$

Since $\frac{i}{2}$ inside the circle $|z| = 2$

$\frac{f(z)}{z-z_0} = \frac{z^3-6}{z-\frac{1}{2}i}$ *then* $z_0 = \frac{1}{2}i,\ f(z) = z^3-6,\ f(z_0) = (\frac{1}{2}i)^3 - 6 = \frac{-i}{8} - 6$ by using

Cauchy's integral Formula

$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\ f(z_0)$ then $\oint_C \frac{z^3-6}{2z-i}\,dz = \frac{1}{2}\oint_C \frac{z^3-6}{z-\frac{1}{2}i}\,dz = \frac{1}{2}(2\pi i)\,f(\frac{1}{2}i) = \frac{\pi}{8} - 6\pi i$

Example (7):

integrate the function $g(z)=\frac{z^2+1}{z^2-1}$ on the curve C in the following cases

(i)where is the circle $|z-1|=1$ (ii) $C:\left|z-\frac{1}{2}\right|=1$

(iii) $C:\left|z-\frac{1}{2}i=1\right|$ (iv) $C:|z+i|=1$

Solution:

Note that the integrant in Cauchy's integral Formula $\frac{f(z)}{z-z_0}$ is nonanalytic at $z=z_0$.

In this example $g(z)=\frac{z^2+1}{z^2-1}$ is non-analytic at $z=\pm 1$ in this case we apply Cauchy's integral Formula where the integral curve is a circle contains the points $z=\pm 1$.

In the cases **(iii)** and **(iv)** $g(z)$ is analytic inside C then $\oint_C \frac{z^2+1}{z^2-1}dz=0$

(i) $\oint_C \frac{z^2+1}{z^2-1}dz=\oint_C \frac{z^2+1}{(z-1)(z+1)}dz$ where C is the circle $|z-1|=1$ which contains

the point $z_0=1$ then $f(z)=\frac{z^2+1}{(z+1)}$ is analytic within and on C an

$f(z_0)=f(1)=\frac{1+1}{(1+1)}=1$ hence

$$\oint_C \frac{z^2+1}{z^2-1}dz=\oint_C \frac{(z^2+1)/(z+1)}{(z-1)}dz=2\pi i f(1)=2\pi i$$

(ii) where C is the circle $C:\left|z+\frac{1}{2}\right|=1$ which contains the point $z_0=-1$ then

$f(z)=\frac{z^2+1}{(z-1)}$, is analytic inside C and

$$\oint_C \frac{z^2+1}{z^2-1}dz = \oint_C \frac{(z^2+1)/(z-1)}{(z+1)}dz = 2\pi i f(-1) = -2\pi i$$

Example (8):

Evaluate $\oint_C \frac{\cos z}{(z-\pi i)^2}\, dz$ on any contour C contains $z = \pi i$.

Solution:

Compare the given integral by (32) we find

$z_0 = \pi i,$

$f(z) = \cos z,$

$n+1 = 2,\ then\ n = 1$

$f'(z) = -\sin z \Rightarrow f'(z_0) = -\sin \pi i = -i \sinh \pi$

$$\oint_C \frac{\cos z}{(z-\pi i)^2}dz = \left[2\pi i(\cos z)'\right]_{z=\pi i} = -2\pi i(\sin \pi i) = -2\pi \sinh \pi$$

Example (9):

Evaluate $\oint_C \frac{z^4-3z^2+6}{(z+i)^3}\, dz$ where C is the circle $|z| = 2$.

Solution:

Since $z = -i$ inside the integral curve C then

$$\oint_C \frac{z^4-3z^2+6}{(z+i)^3}\, dz = \left[\pi i(z^4-3z^2+6)''\right]_{z=-i} = \pi i\left[12z^2-6\right]_{z=-i} = -18\pi i$$

Example (10):

Evaluate $\oint_C \frac{e^z}{(z-1)^2(z^2+4)}dz$ where C is the circle $|z| = \frac{3}{2}$.

Solution:

Since $z=1$ inside the integral curve C and $z=\pm 2i$ outside C then $\dfrac{e^z}{(z^2+4)}$ is analytic inside C and we have

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)}dz = 2\pi i\left(\frac{e^z}{z^2+4}\right)'_{z=1} = 2\pi i\left(\frac{e^z(z^4+4)-e^z.2z}{(z^2+4)^2}\right)_{z=1} = \frac{6e\pi}{25}i$$

Example (11):

Show that $I=\oint_C \dfrac{z^2}{(z^2+4)^2}dz = \dfrac{\pi}{4}$ where C is the circle $x^2+y^2=4y$.

Solution:

C is the circle with center at $(0,2i)$ and radius equal 2 ,since $f(z)$ is analytic on and inside C then we have

$$\frac{z^2}{(z^2+4)^2} = \frac{z^2}{(z+2i)^2(z-2i)^2} = \frac{\left(z^2/(z+2i)^2\right)}{(z-2i)^2} = \frac{f(z)}{(z-2i)^2}$$

$$I=\oint_C \frac{z^2dz}{(z^2+4)^2} = \oint_C \frac{f(z)}{(z-2i)^2}dz$$ from Cauchy formula we have

$$f'(2i) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-2i)^2}dz = \frac{I}{2\pi i}$$

$$f'(z) = \left[\left(\frac{z}{z+2i}\right)\left(\frac{z}{z+2i}\right)\right]' = 2\left(\frac{z}{z+2i}\right)\left(\frac{\not{z}+2i-\not{z}}{(z+2i)^2}\right) = \frac{4iz}{(z+2i)^3}$$

$$f'(2i) = \frac{-8}{-64i} = \frac{1}{8i} \qquad \therefore I = 2\pi i f'(2i) = 2\pi i\left(\frac{1}{8i}\right) = \frac{\pi}{4}$$